

Interactive Complex Analysis: Supplementing the Traditional Course with Computer Visualization*

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Abstract

We present an applet and learning activities used to aid in the visualization and understanding of properties of complex-valued functions. We discuss how the technology can be integrated into a traditional Complex Variables class via exploration, discussion, and regular use.

1 Introduction

Complex Variables is an inherently geometric subject. However, it is often taught in an entirely analytic manner because much of the beautiful and rich geometric structure of the field is not easily visualized. Instructors often limit themselves to drawing the most basic pictures of mapping properties of functions because of the four-dimensional nature of the pictures. It is easy to draw images of lines and circles for the most elementary functions, but other objects can get complicated very quickly. Here we describe a freely available applet, *ComplexTool*, along with some exploratory questions and activities. *ComplexTool* combines the graphical abilities of many pre-existing tools, including some other Java applets, while adding a very intuitive user interface. We have used this applet in a complex variables class and feel that it augments the course very well. Our experiences reflected here are based on a senior-level traditional complex variables class, primarily taken by mathematics majors but open to any student who has taken an advanced differential equations class. We have now used *ComplexTool* in four offerings of the course with a total of approximately fifty students.

2 Technical Implementation

ComplexTool is written in Java and will run on any computer platform. The applet, its version history, and our exploratory exercises can be run and/or downloaded (for free) from

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<http://jimrolf.com/java/complexTool/complexTool.html>. *ComplexTool* was programmed by Rolf with mathematical guidance from Schaubroeck for the express purpose of visualizing complex-valued functions. It is part of a larger NSF-funded project to create applets for visualizations of many complex analysis topics. A text using a suite of applets, authored by experts from four different academic institutions, is forthcoming. Readers are encouraged to contact Rolf for more information about the technical aspects of *ComplexTool*, to report bugs, and/or to suggest additional features.

3 Description of *ComplexTool*

ComplexTool is designed to facilitate easy and quick input of complex-valued functions and, with the click of a button, to see the result of this function applied to various domains. Domain options include any polar region (see Figure ??), any rectangular region (see Figure ??), or none at all (see Figure ??). Each of these options allows the student to sketch in the domain with the mouse and, in real time, see the transformation applied to the sketch. The user can also select polar and rectangular domains with the mouse and move them around in order to see the effects of the transformation in the range. When using *ComplexTool*, one does have to remember that it is designed primarily to help the user visualize geometric ideas in complex analysis. It does not compute derivatives, integrals, residues, partial fractions, and the like. But we believe that the simplicity of the applet is the very thing that encourages and enables explorations that are useful in a beginning Complex Variables class.

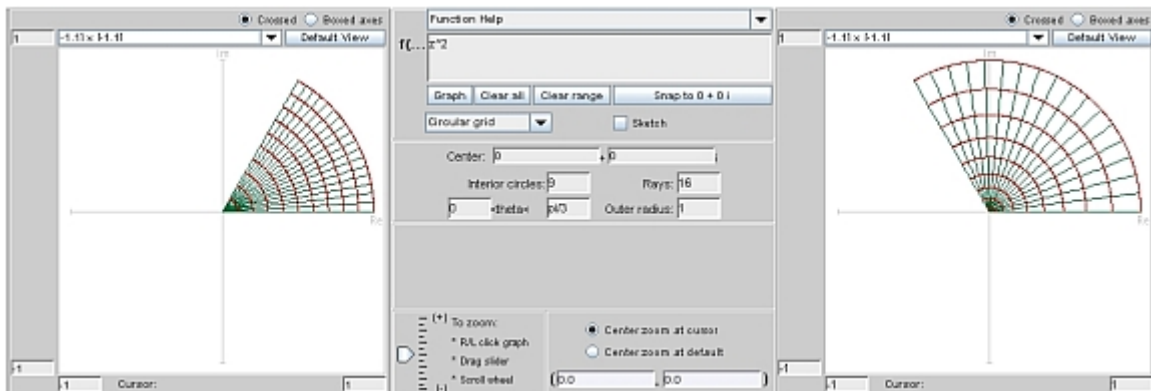


Figure 1: The function $f(z) = z^2$. The domain is on the left and the range is on the right.

4 Independent Exploration and Class Discussion

We believe that handing a technological tool to students with little guidance may result in little learning value. So when we used this applet in a senior-level class for mathematics majors, we first briefly demonstrated it class, and then gave the students a short assignment to use the applet to explore functions. (The full assignment can be found at <http://jimrolf.com/java/complexTool/complexTool.html>.) Our goal with these explorations was that students would not only develop geometric intuition, but that this intuition would also lead to deeper understanding of the analytic properties causing the observed

transformations. In the short assignment we required students to investigate the mapping properties of the two functions $f(z) = z^2$ and $f(z) = e^z$ with minimal suggestions to guide their explorations.

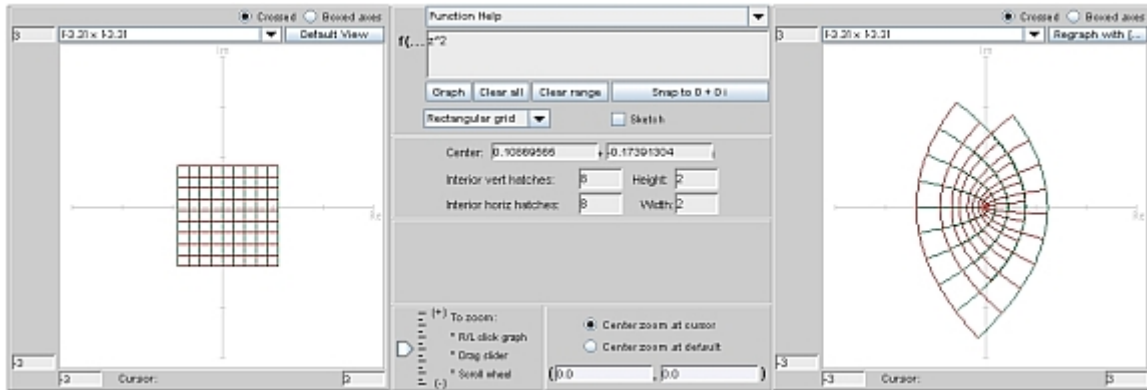


Figure 2: The image of a rectangular grid under the function $f(z) = z^2$. Note the overlap in the range, showing that the function is not one-to-one.

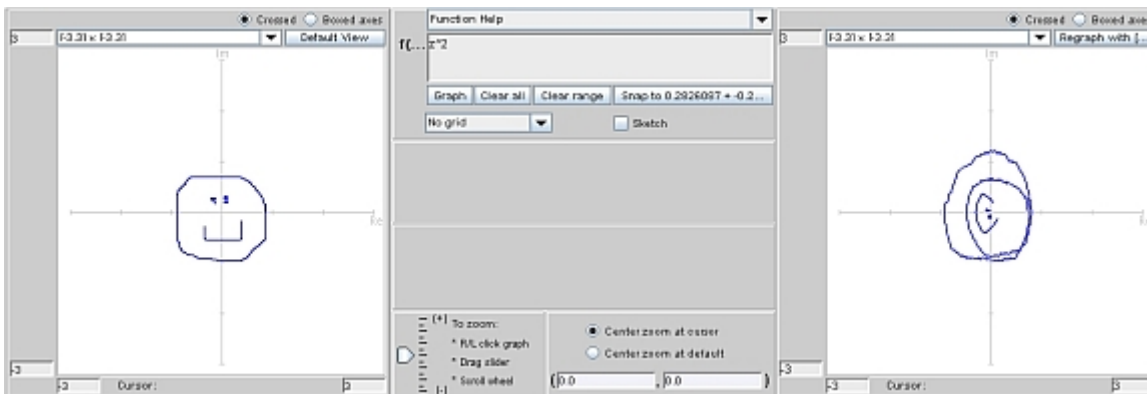


Figure 3: The image of a smiley face under the function $f(z) = z^2$, using the sketching tool.

For example, we suggested that students examine the squaring function by dragging a rectangular grid into each quadrant of the domain and describe how the squaring function mapped this rectangular grid into the range during the dragging operation. We also had the students make a portion of a polar grid, centered at the origin, and describe its image under the squaring function. For the exponential function, we asked the students to describe the images of horizontal and vertical lines as a rectangular grid was dragged towards each corner of the visible domain. The students then wrote a short paper summarizing these results along with other explorations.

The goal of the exploration was achieved—the students developed some geometric intuition and also analyzed the equations to create proofs of their conjectures about the functions. Details are given below.

Exploration 4.1: $f(z) = z^2$. Many of the students examined the squaring function using the representation $z = re^{i\theta}$. This enabled them to notice the angle-doubling property of z^2 . During the class discussion following these explorations, those who instead chose only to use the representation $z = x + iy$ suddenly became very interested in understanding polar coordinates to understand this property! The frustrations they encountered while using a less-helpful representation for z during explorations elicited an desire to use another representation for the complex variable z .

Exploration 4.2: $f(z) = e^z$. The students accurately described the properties of the exponential function using rectangular coordinates, and all noticed its periodic nature. A typical student exploration of the exponential function started with the default square grid. All of the students noticed that the square grid seemed to be turned into a polar grid, with the vertical lines (green) getting mapped to circles and the horizontal lines (red) getting mapped to rays (see Figure ??). Moreover, the students would drag that grid along the imaginary axis, and notice that its image would spin around the origin, prompting the student to explore the periodic nature of the exponential function.

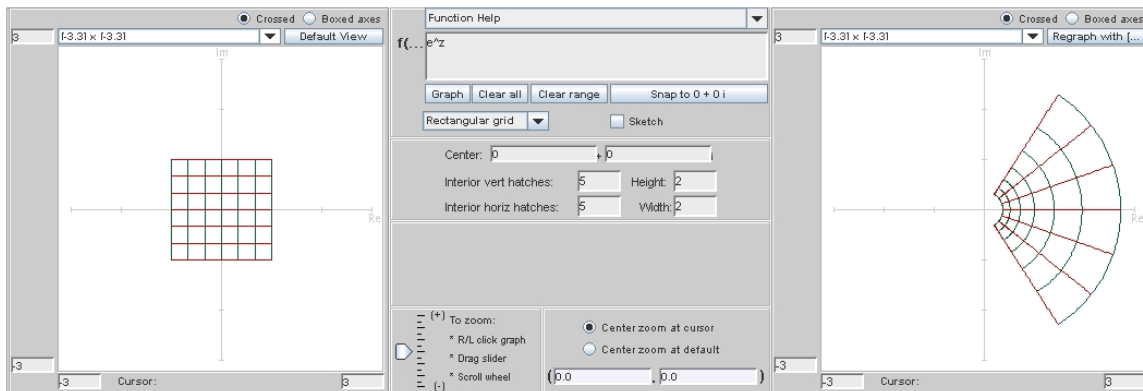


Figure 4: The image of a rectangular grid under the exponential function.

These preliminary explorations done by the students not only helped the students learn about particular functions, but also aided the in-class discussion. When we started to rigorously define these functions in class, students seemed to more easily understand the abstract mathematical properties and associated notation because of their strong intuitive understanding they had gained via geometric explorations.

Exploration 4.3: Student-chosen Functions. In addition to exploring prescribed functions, students were asked to mimic these explorations with a non-linear function of their choosing. It was in these explorations that we saw the most student engagement. The functions that the students chose to explore included \sqrt{z} , $1/z$, z^3 , and $\log(z)$. The mapping properties of a disk under the reciprocal function was described by one student by, “Anything inside the circle will be mapped outside, and anything outside will be mapped inside.” Though this statement is true only for circles of radius 1, the student had found a fundamental property of the inverse function—inversion through the unit circle. Another student noted that, “a line maps to a circle that touches (but excludes, obviously) the origin, and vice versa.” These observations opened the way for discussions of the mapping properties of linear fractional transformations later in the semester.

Exploration 4.4: $f(z) = \log(z)$. Many students chose to examine the inverse of the exponential function, $\log(z)$. Since they had noticed that the exponential function turned a rectangular grid into a polar grid, they correctly conjectured that the logarithmic function should turn a polar grid into a rectangular one. However, students encountered behavior that appeared to be very strange due to the branch cut, which had not yet been discussed in class. Of the students who investigated this function, most expressed some confusion about what was really happening, especially for inputs with a negative real part. This begged for in-class discussion, which we began immediately upon reading the papers. In particular, we demonstrated that the image of a polar grid “breaks up” as it crosses the negative real axis, as shown in figure ??.

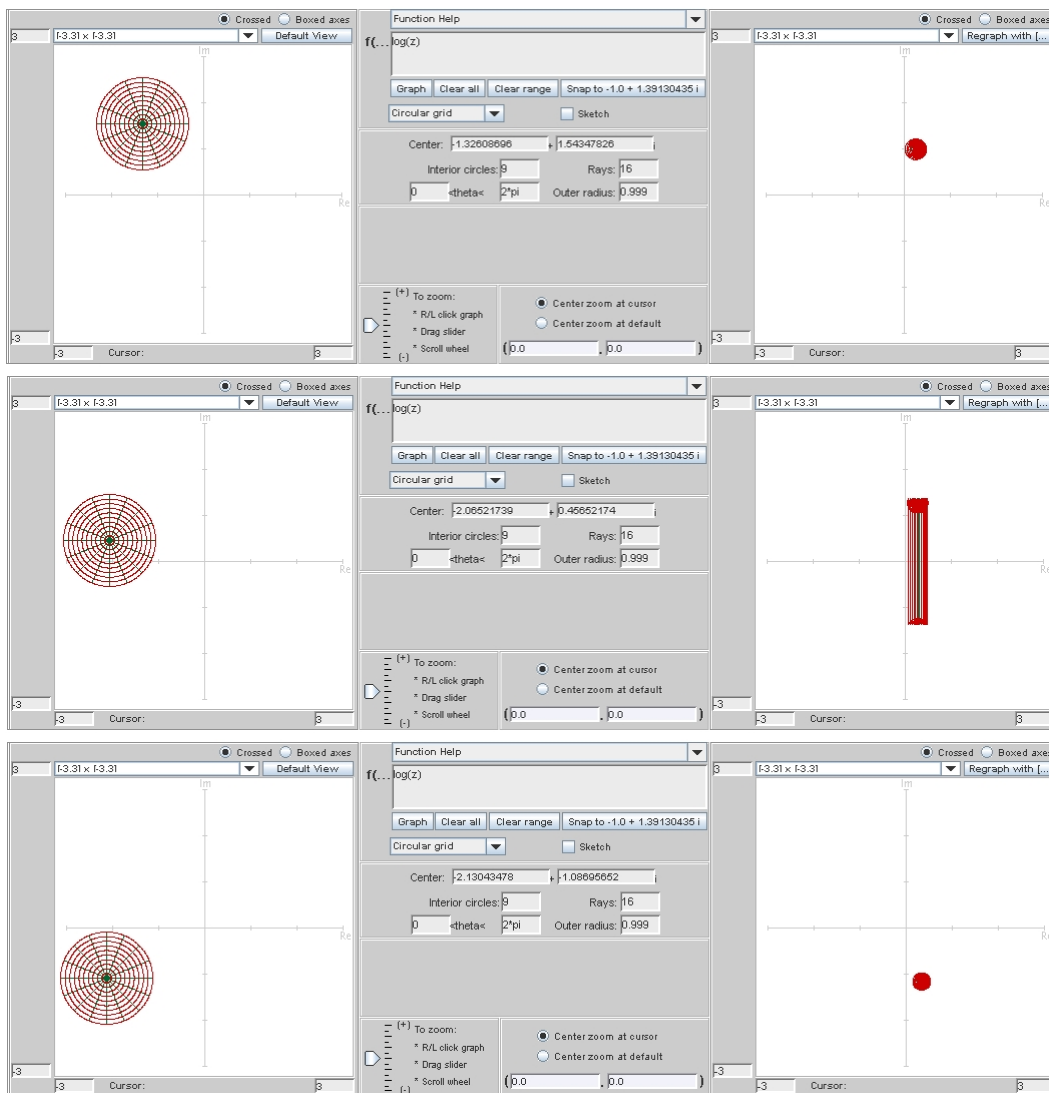


Figure 5: A sequence of pictures showing the image under $f(z) = \log(z)$ when “dragging” the domain circle across the negative real axis. The long vertical lines in the middle figure are due to the applet “connecting the dots.”

We found that the follow-on discussion in class after student use of *ComplexTool* produced invaluable learning moments. It gave students an opportunity to learn from each other's questions and insights and gave the instructor an opportunity to highlight key points. For example, the in-class discussion about the squaring function naturally led to a student's exploration about the cubing function, giving that student an opportunity to share what he had learned with the class. The in-class discussion about the logarithm gave a good base of knowledge that we called upon later in the semester with the study of branch cuts. In fact, the introduction to branch cuts started with opening the applet again and repeating the strange behavior around the negative real axis of both the logarithm and the square root functions. The students had their memories jogged with comments like "Remember when we were confused about what was going on here? Well, now we'll learn the mathematics behind those strange pictures!"

5 Independent learning via the applet

Another use of the applet was to encourage students to "discover" a theorem instead of having the teacher or textbook introduce the theorem. This desire led to another exploration exercise, in which we asked students to conjecture what kinds of functions would preserve angles. We were hoping that the students would discover for themselves the following theorem: An analytic function $f(z)$ is conformal (preserving the magnitude and direction of angles) at all points where $f'(z) \neq 0$. (see Saff & Snider, 1993; Zill & Shanahan, 2003, for example.) The students were given a short list of functions, some of which were polynomials in z and others that were polynomials in \bar{z} . We asked students to graph the images of various grids under these functions, zoom in on points of intersection of the gridlines, and determine if the functions preserved angles. Apart from these directions, we gave them no other instructions. While we intentionally provided minimal guidance for exploration, we were uncertain if students would take the time to genuinely explore enough to address their own conjectures. We were thrilled to see that all of the students did make the initial conjecture that analytic functions generally preserve angles (see figures ??), while non-analytic functions may not preserve angles (figure ??). Several students were inspired enough to take the extra step to look up the result in the textbook that analytic functions preserve angles, but only at those places where the derivative is nonzero.

After the students had completed this activity, we again followed up in class with a brief discussion. Though the independent exploration did not address the sense (direction) of angles, this property also can be explored using the sketching function of *ComplexTool*. By tracing near one angle, and looking at the image of the sketched line, we see that the function $f(z) = \bar{z}$ does indeed preserve the magnitude of angles, but reverses the direction. Since students had discovered the basic principle of angle-preservation for themselves, the discussion centered on the subtler points of sense of an angle and of what happens when the derivative of an analytic function is not zero (see figures ?? and ??). In short we saw students were motivated to know why analytic functions are conformal. This motivation seemed greater than student motivation in prior semesters when we required them to simply memorizing basic facts about conformality.

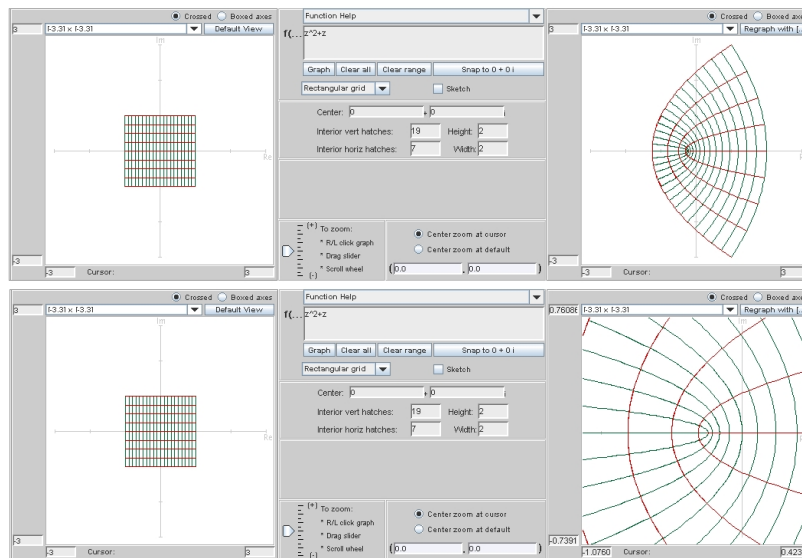


Figure 6: The image of a square under $f(z) = z^2 + z$. When zooming in on the image of $z = -\frac{1}{2}$, to where $f(z) = -\frac{1}{4}$, one can see the bending of the vertical lines, showing that angles are not preserved at $z = -\frac{1}{2}$. Note that other intersections are at right angles, showing the angle preservation elsewhere.

6 Regular In-Class Use

We also used *ComplexTool* in class on a regular basis. In the middle of discussions about rigorous topics, students frequently would generate pictures in *ComplexTool*. The applet also provided a vehicle for students to make conjectures about mapping properties of functions, and to quickly check whether the conjecture appeared to be true.

Exploration 6.1: Trigonometric Functions. After rigorously developing the mapping properties of the sine function, we would ask the students to conjecture about the mapping properties of the cosine function, and then test the conjecture immediately. For example, we developed the fact that the sine function maps vertical and horizontal lines to hyperbolas and ellipses, respectively. The students then understood that the complex cosine probably did the same thing, but with a horizontal shift of $\pi/2$. Figures ?? and ?? demonstrate this relationship, applying the shift in the domain.

Students also frequently used the applet to check their work. A common exercise in complex variables is to have students create a linear function that maps one region of the plane onto another region. When armed with *ComplexTool*, students would routinely build the function as in any other complex variables class, but also follow up that analysis by checking their result using the applet. It had been our experience in teaching without the technology that students would often not check their final result, but having the applet available to quickly check their work seemed to encourage more critical thinking about their answers.

The daily use of the applet certainly did not supercede rigorous development of the mapping properties of functions, but it was very useful in building intuition. For example, in the discussion of the complex trigonometric functions, it was very easy to demonstrate that the fundamental region $-\frac{\pi}{2} \leq \text{Re}(z) < \frac{\pi}{2}$ appears to map onto the plane under the sine function, but the geometric demon-

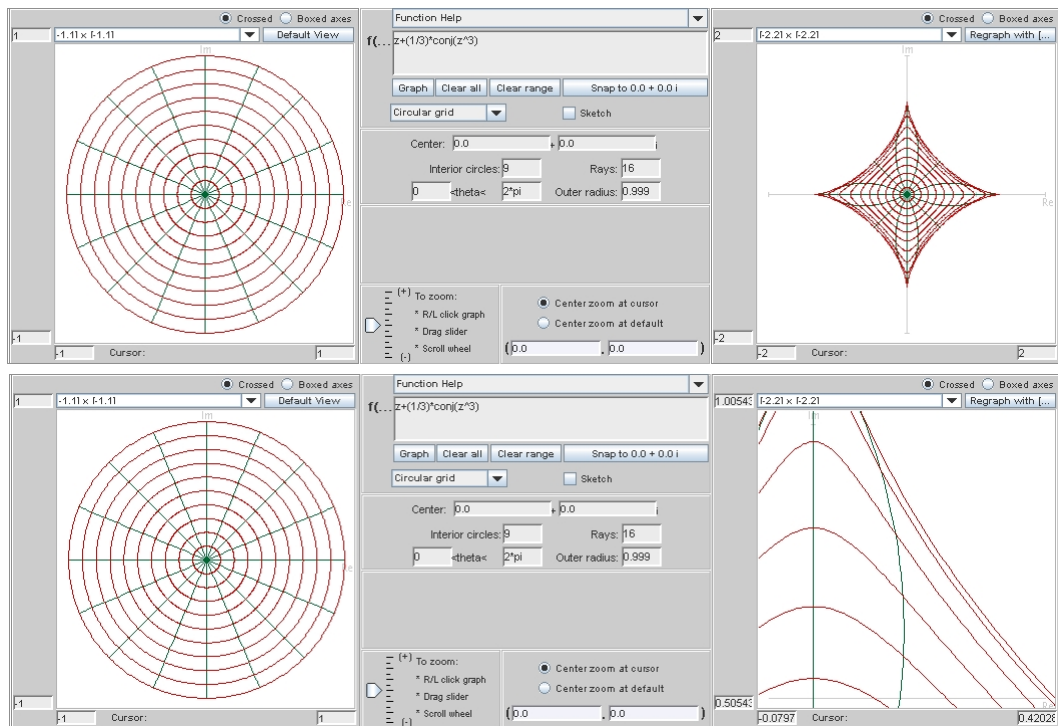


Figure 7: The image of the unit disk under $f(z) = z + \frac{z^3}{3}$. The second figures zooms in on the first quadrant to show that angles are not preserved. Note that the images should be at right angles, since the circles and rays intersect at right angles.

stration by no means proved that fact (see figure ??). We feel that the ease of quickly examining conjectures enriches the classroom experience, and it is useful for students to understand the role that technology can play in mathematics. In particular, having technology available but always following up with rigorous analysis seemed to help the students not just “trust” the computer answers.

7 Conclusion

The use of *ComplexTool* in a complex variables class had a positive impact on student participation in the class. The students found the interface easy to use, and thus frequently worked with the applet to try to understand the mapping properties of functions. Anecdotally, we saw that the students had a better geometric understanding of complex functions than in classes that we taught without *ComplexTool*. In fact, one student wrote, “Without it I don’t think I would have nearly the same understanding of how the complex mappings work.” Students seemed to have a good grasp of the mapping properties of basic functions, and had the confidence that they had a tool to help them understand mapping properties of more complicated functions. We believe that the exploration exercises, as well as frequent in-class use, enhanced the students’ geometric understanding of the material and this intuition provided greater motivation to do the mathematical analysis to test conjectures and analyze results.

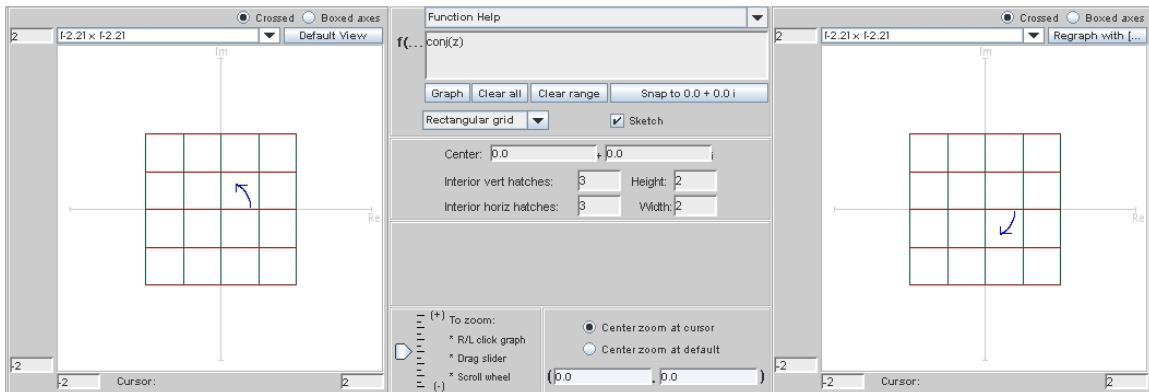


Figure 8: Under the function $f(z) = \bar{z}$, the angles are preserved but direction is reversed.

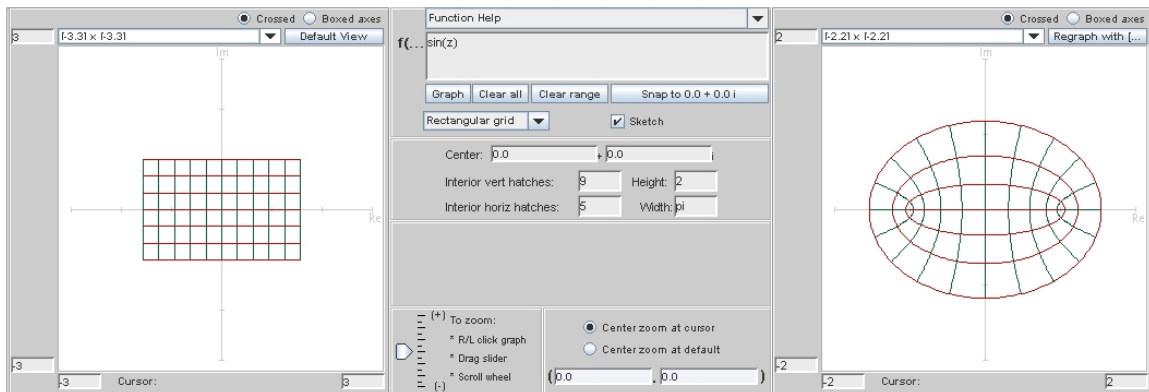


Figure 9: The image of a rectangular region under $f(z) = \sin(z)$.

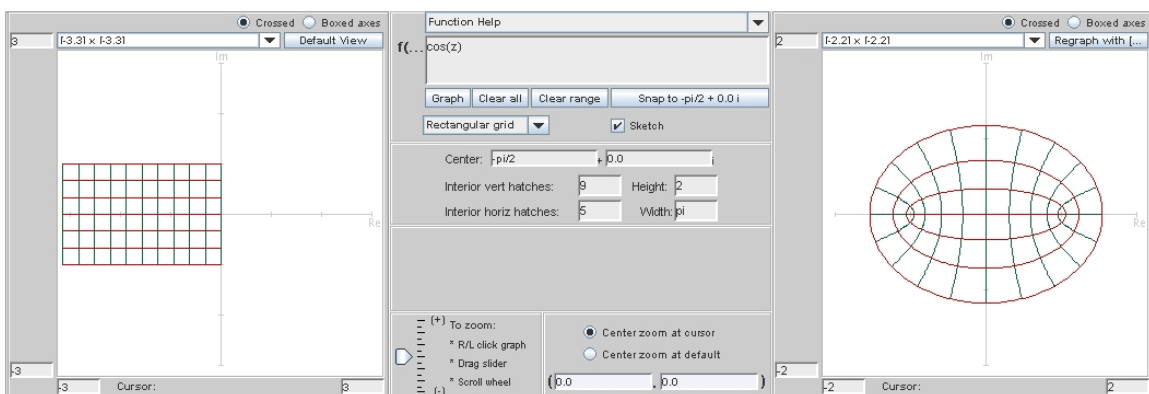


Figure 10: The image of a shifted rectangular region under $f(z) = \cos(z)$.

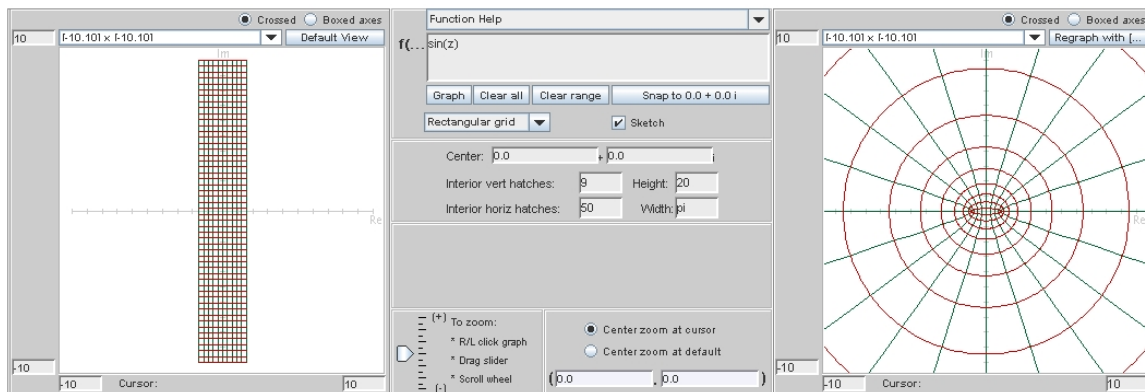


Figure 11: Under the action of sine, the vertical strip of width π appears to cover the plane.

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